FOURTH EDITION

CALCULUS

EARLY TRANSCENDENTALS

Jon Rogawski ▪ Colin Adams ▪ Robert Franzosa
University of California, Los Angeles · Williams College · The University of Maine

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TO ALEXA AND COLTON –Colin
TO MY FAMILY –Bob

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ABOUT THE AUTHORS

Jon Rogawski

As a successful teacher for more than 30 years, Jon Rogawski listened and learned much from his own students. These valuable lessons made an impact on his thinking, his writing, and his shaping of a calculus text.

Jon Rogawski received his undergraduate and master's degrees in mathematics simultaneously from Yale University, and he earned his PhD in mathematics from Princeton University, where he studied under Robert Langlands. Before joining the Department of Mathematics at UCLA in 1986, where he was a full professor, he held teaching and visiting positions at the Institute for Advanced Study, the University of Bonn, and the University of Paris at Jussieu and Orsay.

Jon's areas of interest were number theory, automorphic forms, and harmonic analysis on semisimple groups. He published numerous research articles in leading mathematics journals, including the research monograph *Automorphic Representations of Unitary Groups in Three Variables* (Princeton University Press). He was the recipient of a Sloan Fellowship and an editor of the *Pacific Journal of Mathematics* and the *Transactions of the AMS*.

Sadly, Jon Rogawski passed away in September 2011. Jon's commitment to presenting the beauty of calculus and the important role it plays in students' understanding of the wider world is the legacy that lives on in each new edition of *Calculus*.

Colin Adams

Colin Adams is the Thomas T. Read professor of Mathematics at Williams College, where he has taught since 1985. Colin received his undergraduate degree from MIT and his PhD from the University of Wisconsin. His research is in the area of knot theory and low-dimensional topology. He has held various grants to support his research and written numerous research articles.

Colin is the author or co-author of *The Knot Book, How to Ace Calculus: The Streetwise Guide, How to Ace the Rest of Calculus: The Streetwise Guide, Riot at the Calc Exam and Other Mathematically Bent Stories, Why Knot?, Introduction to Topology: Pure and Applied,* and *Zombies & Calculus.* He co-wrote and appears in the videos "The Great Pi vs. e Debate" and "Derivative vs. Integral: the Final Smackdown."

He is a recipient of the Haimo National Distinguished Teaching Award from the Mathematical Association of America (MAA) in 1998, an MAA Polya Lecturer for 1998-2000, a Sigma Xi Distinguished Lecturer for 2000-2002, and the recipient of the Robert Foster Cherry Teaching Award in 2003.

Colin has two children and one slightly crazy dog, who is great at providing the entertainment.

Robert Franzosa

Robert (Bob) Franzosa is a professor of mathematics at the University of Maine where he has been on the faculty since 1983. Bob received a BS in mathematics from MIT in 1977 and a PhD in mathematics from the University of Wisconsin in 1984. His research has been in dynamical systems and in applications of topology in geographic information systems. He has been involved in mathematics education outreach in the state of Maine for most of his career.

Bob is a co-author of *Introduction to Topology: Pure and Applied* and *Algebraic Models in Our World.* He was awarded the University of Maine's Presidential Outstanding Teaching award in 2003.

Bob is married, has two children, three step-children, and one grandson.

SECTION 2.1 was largely rewritten so that its focus is on motivating the need for limits via the concepts of velocity and the tangent line. The content on rate of change that did not treat velocity was moved elsewhere.

SECTION 4.1 was rewritten and reorganized to clarify the relationship between the different types of linear approximation. In particular, we wanted to reinforce the understanding that the various types of linear approximation are all based on the idea that the tangent line approximates the curve close to the point of tangency.

SECTION 8.1 The section on probability was moved from Section 7.8 to 8.1 so that it appears in the chapter on applications of the integral rather than the chapter on techniques of integration.

Chapter 1: Precalculus Review 1

1.1 Real Numbers, Functions, and Graphs 1
1.2 Linear and Quadratic Functions 14
1.3 The Basic Classes of Functions 22
1.4 Trigonometric Functions 27
1.5 Inverse Functions 35
1.6 Exponential and Logarithmic Functions 44
1.7 Technology: Calculators and Computers 53
Chapter Review Exercises 58

Chapter 2: Limits 61

2.1 The Limit Idea: Instantaneous Velocity and Tangent Lines 61
2.2 Investigating Limits 67
2.3 Basic Limit Laws 76
2.4 Limits and Continuity 79
2.5 Indeterminate Forms 90
2.6 The Squeeze Theorem and Trigonometric Limits 96
2.7 Limits at Infinity 101
2.8 The Intermediate Value Theorem 107
2.9 The Formal Definition of a Limit 112
Chapter Review Exercises 119

Chapter 3: Differentiation 123

3.1 Definition of the Derivative 123
3.2 The Derivative as a Function 132
3.3 Product and Quotient Rules 146
3.4 Rates of Change 153
3.5 Higher Derivatives 163
3.6 Trigonometric Functions 168
3.7 The Chain Rule 173
3.8 Implicit Differentiation 181
3.9 Derivatives of General Exponential and Logarithmic Functions 190
3.10 Related Rates 198
Chapter Review Exercises 206

Chapter 4: Applications of the Derivative 211

4.1 Linear Approximation and Applications 211
4.2 Extreme Values 220
4.3 The Mean Value Theorem and Monotonicity 230
4.4 The Second Derivative and Concavity 238
4.5 L'Hôpital's Rule 246
4.6 Analyzing and Sketching Graphs of Functions 254
4.7 Applied Optimization 263
4.8 Newton's Method 276
Chapter Review Exercises 282

Chapter 5: Integration 285

5.1 Approximating and Computing Area 285
5.2 The Definite Integral 298
5.3 The Indefinite Integral 309
5.4 The Fundamental Theorem of Calculus, Part I 318
5.5 The Fundamental Theorem of Calculus, Part II 325
5.6 Net Change as the Integral of a Rate of Change 333
5.7 The Substitution Method 339
5.8 Further Integral Formulas 347
Chapter Review Exercises 352

Chapter 6: Applications of the Integral 357

6.1 Area Between Two Curves 357
6.2 Setting Up Integrals: Volume, Density, Average Value 365
6.3 Volumes of Revolution: Disks and Washers 376
6.4 Volumes of Revolution: Cylindrical Shells 385
6.5 Work and Energy 392
Chapter Review Exercises 398

Chapter 7: Techniques of Integration 401

7.1 Integration by Parts 401
7.2 Trigonometric Integrals 407
7.3 Trigonometric Substitution 415
7.4 Integrals Involving Hyperbolic and Inverse Hyperbolic Functions 421
7.5 The Method of Partial Fractions 426
7.6 Strategies for Integration 435
7.7 Improper Integrals 442
7.8 Numerical Integration 453
Chapter Review Exercises 463

Chapter 8: Further Applications of the Integral 467

8.1 Probability and Integration 467
8.2 Arc Length and Surface Area 473
8.3 Fluid Pressure and Force 480
8.4 Center of Mass 486
Chapter Review Exercises 496

Chapter 9: Introduction to Differential Equations 499

9.1 Solving Differential Equations 499
9.2 Models Involving $y' = k(y - b)$ 509
9.3 Graphical and Numerical Methods 516
9.4 The Logistic Equation 523
9.5 First-Order Linear Equations 528
Chapter Review Exercises 534

Chapter 10: Infinite Series 537

10.1 Sequences 537
10.2 Summing an Infinite Series 548

THE PREVIOUS EDITION'S SECTION 5.9 was eliminated because it appeared in the main introductory chapter on integration yet had little content involving the integral. The main topics from the section were placed elsewhere where they fit better. For example, the material on differential equations and on exponential growth and decay was moved to the chapter on differential equations.

SECTION 9.1 was rewritten to provide a more-straightforward introduction to differential equations and methods of solving them. Furthermore we wrote a few new examples that replaced a rather technical derivation to provide for a wider variety of simpler, more accessible application examples.

SECTION 14.4 The development of the concept of differentiability in Section 14.4 was rewritten to provide a clearer pathway from the basic idea of the existence of partial derivatives to the more-technical notion of differentiability. We dropped the concept of local linearity introduced in previous editions because it is redundant and adds an extra layer of technical detail that can be avoided.

CONTENTS

10.3 Convergence of Series with Positive Terms 560
10.4 Absolute and Conditional Convergence 569
10.5 The Ratio and Root Tests and Strategies for Choosing Tests 575
10.6 Power Series 580
10.7 Taylor Polynomials 592
10.8 Taylor Series 603
Chapter Review Exercises 615

Chapter 11: Parametric Equations, Polar Coordinates, and Conic Sections 619

11.1 Parametric Equations 619
11.2 Arc Length and Speed 631
11.3 Polar Coordinates 637
11.4 Area and Arc Length in Polar Coordinates 646
11.5 Conic Sections 651
Chapter Review Exercises 665

Chapter 12: Vector Geometry 669

12.1 Vectors in the Plane 669
12.2 Three-Dimensional Space: Surfaces, Vectors, and Curves 680
12.3 Dot Product and the Angle Between Two Vectors 690
12.4 The Cross Product 700
12.5 Planes in 3-Space 712
12.6 A Survey of Quadric Surfaces 719
12.7 Cylindrical and Spherical Coordinates 727
Chapter Review Exercises 734

Chapter 13: Calculus of Vector-Valued Functions 737

13.1 Vector-Valued Functions 737
13.2 Calculus of Vector-Valued Functions 744
13.3 Arc Length and Speed 754
13.4 Curvature 760
13.5 Motion in 3-Space 772
13.6 Planetary Motion According to Kepler and Newton 780
Chapter Review Exercises 787

Chapter 14: Differentiation in Several Variables 789

14.1 Functions of Two or More Variables 789
14.2 Limits and Continuity in Several Variables 800
14.3 Partial Derivatives 808
14.4 Differentiability, Tangent Planes, and Linear Approximation 818
14.5 The Gradient and Directional Derivatives 827
14.6 Multivariable Calculus Chain Rules 840
14.7 Optimization in Several Variables 850
14.8 Lagrange Multipliers: Optimizing with a Constraint 865
Chapter Review Exercises 875

Chapter 15: Multiple Integration 879

15.1 Integration in Two Variables 879
15.2 Double Integrals over More General Regions 891
15.3 Triple Integrals 905
15.4 Integration in Polar, Cylindrical, and Spherical Coordinates 917
15.5 Applications of Multiple Integrals 927
15.6 Change of Variables 939
Chapter Review Exercises 953

Chapter 16: Line and Surface Integrals 957

16.1 Vector Fields 957
16.2 Line Integrals 967
16.3 Conservative Vector Fields 983
16.4 Parametrized Surfaces and Surface Integrals 995
16.5 Surface Integrals of Vector Fields 1009
Chapter Review Exercises 1020

Chapter 17: Fundamental Theorems of Vector Analysis 1023

17.1 Green's Theorem 1023
17.2 Stokes' Theorem 1037
17.3 Divergence Theorem 1049
Chapter Review Exercises 1061

Appendices A1

A. The Language of Mathematics A1
B. Properties of Real Numbers A7
C. Induction and the Binomial Theorem A12
D. Additional Proofs A16

ANSWERS TO ODD-NUMBERED EXERCISES ANS1

REFERENCES R1

INDEX I1

Additional content can be accessed online at www.macmillanlearning.com/calculuset4e:

Additional Proofs:

L'Hôpital's Rule
Error Bounds for Numerical Integration
Comparison Test for Improper Integrals

Additional Content:

Second-Order Differential Equations
Complex Numbers

SECTION 10.7 We have chosen a somewhat traditional location for the section on Taylor polynomials, placing it directly before the section on Taylor series in Chapter 10. We feel that this placement is an improvement over the previous edition where the section was isolated in a chapter that primarily was about applications of the integral. The subject matter in the Taylor polynomials section works well as an initial step toward the important topic of Taylor series representations of specific functions. The Taylor polynomials section can serve as a follow-up to linear approximation in Section 4.1. Consequently, Taylor polynomials (except for Taylor's Theorem at the end of the section, which involves integration) can be covered at any point after Section 4.1.

CALCULUS: EARLY TRANSCENDENTALS, FOURTH EDITION

On Teaching Mathematics

We consider ourselves very lucky to have careers as teachers and researchers of mathematics. Through many years (over 30 each) teaching and learning mathematics we have developed many ideas about how best to present mathematical concepts and to engage students working with and exploring them. We see teaching mathematics as a form of storytelling, both when we present in a classroom and when we write materials for exploration and learning. The goal is to explain to students in a captivating manner, at the right pace, and in as clear a way as possible, how mathematics works and what it can do for them. We find mathematics to be intriguing and immensely beautiful. We want students to feel that way, too.

On Writing a Calculus Text

It has been an exciting challenge to author the recent editions of Jon Rogawski's calculus book. We both had experience with the early editions of the text and had a lot of respect for Jon's approach to them. Jon's vision of what a calculus book could be fits very closely with our own. Jon believed that as math teachers, how we present material is as important as what we present. Although he insisted on rigor at all times, he also wanted a book that was clearly written, that could be read by a calculus student and would motivate them to engage in the material and learn more. Moreover, Jon strived to create a text in which exposition, graphics, and layout would work together to enhance all facets of a student's calculus experience. Jon paid special attention to certain aspects of the text:

1. Clear, accessible exposition that anticipates and addresses student difficulties.
2. Layout and figures that communicate the flow of ideas.
3. Highlighted features that emphasize concepts and mathematical reasoning including Conceptual Insight, Graphical Insight, Assumptions Matter, Reminder, and Historical Perspective.
4. A rich collection of examples and exercises of graduated difficulty that teach basic skills as well as problem-solving techniques, reinforce conceptual understanding, and motivate calculus through interesting applications. Each section also contains exercises that develop additional insights and challenge students to further develop their skills.

Our approach to writing the recent editions has been to take the strong foundation that Jon provided and strengthen it in two ways:

- To fine-tune it, while keeping with the book's original philosophy, by enhancing presentations, clarifying concepts, and emphasizing major points where we felt such adjustments would benefit the reader.
- To expand it slightly, both in the mathematics presented and the applications covered. The expansion in mathematics content has largely been guided by input from users and reviewers who had good suggestions for valuable additions (for example, a section on how to decide which technique to employ on an integration problem). The original editions of the text had very strong coverage of applications in physics and engineering; consequently, we have chosen to add examples that provide applications in the life and climate sciences.

We hope our experience as mathematicians and teachers enables us to make positive contributions to the continued development of this calculus book. As mathematicians, we want to ensure that the theorems, proofs, arguments, and derivations are correct and are presented with an appropriate level of rigor. As teachers, we want the material to be accessible and written at the level of a student who is new to the subject matter. Working from the strong foundation that Jon set, we have strived to maintain the level of quality of the previous editions while making the changes that we believe will bring the book to a new level.

What's New in the Fourth Edition

In this edition we have continued the themes introduced in the third edition and have implemented a number of new changes.

A Focus on Concepts

We have continued to emphasize conceptual understanding over the memorization of formulas. Memorization can never be completely avoided, but it should play a minor role in the process of learning calculus. Students will remember how to apply a procedure or technique if they see the logical progression of the steps in the proof that generates it. And they then understand the underlying concepts rather than seeing the topic as a black box. To further support conceptual understanding of calculus, we have added a number of new Graphical and Conceptual Insights through the book. These include insights that discuss:

- The differences between the expressions “undefined,” “does not exist,” and “indeterminate” in Section 2.5 on indeterminate forms,
- How measuring angles in radians is preferred in calculus over measuring in degrees because the resulting derivative formulas are simpler (in Section 3.6 on derivative rules of trigonometric functions),
- How the Fundamental Theorem of Calculus (Part II) guarantees the existence of an antiderivative for continuous functions (in Section 5.5 on the Fundamental Theorem of Calculus, Part II),
- How the volume-of-revolution formulas in Section 6.3 are special cases of the main volume-by-slices approach in Section 6.2,
- The relationships between a curve, parametrizations of it, and arc length computed from a parametrization (in Section 11.2 on arc length and speed),
- The relationship between linear approximation in multivariable calculus (in Section 14.4) and linear approximation for a function of one variable in Section 4.1.

Simplified Derivations

We simplified a number of derivations of important calculus formulas. These include:

- The derivative rule for the exponential function in Section 3.2,
- The formula for the area of a surface of revolution in Section 8.2,
- The vector-based formulas for lines and planes in 3-space in Sections 12.2 and 12.5.

New Examples in the Life and Climate Sciences

Expanding on the strong collection of applications in physics and engineering that were already in the book, we added a number of applications from other disciplines, particularly in the life and climate sciences. These include:

- The rate of change of day length in Section 3.7
- The log-wind profile in Section 3.9
- A grid-connected energy system in Section 5.2
- A glacier height differential-equations model in Section 9.1
- A predator-prey interaction in Section 11.1
- Geostrophic wind flow in Section 14.5
- Gulf Stream heat flow in Section 15.1

An Introduction to Calculus

In previous editions of the text, the first mathematics material that the reader encountered was a review of precalculus. We felt that a brief introduction to calculus would be a more meaningful start to this important body of mathematics. We hope that it provides the reader with a motivating glimpse ahead and a perspective on why a review of precalculus is a beneficial way to begin.

Additional Historical Content

Historical Perspectives and margin notes have been a well-received feature of previous editions. We added to the historical content by including a few new margin notes about past and contemporary mathematicians throughout the book. For example, we added a margin note in Section 3.1 about the contributions of Sir Isaac Newton and Gottfried Wilhelm Leibniz to the development of calculus in the seventeenth century, and a margin note in Section 12.2 about recent Field’s medalist Maryam Mirzakhani.

New Examples, Figures, and Exercises

Numerous examples and accompanying figures have been added to expand on the variety of applications and to clarify concepts. Figures marked with a DF icon have been made dynamic and can be accessed via WebAssign Premium. A selection of these figures also includes brief tutorial videos explaining the concepts at work.

A variety of exercises have also been added throughout the text, particularly following up on new examples in the sections. The comprehensive section exercise sets are closely coordinated with the text. These exercises vary in difficulty from routine to moderate as well as more challenging. Specialized exercises are identified by icons. For example, indicates problems that require the student to give a written response. There also are icons for problems that require the use of either graphing-calculator technology GU or more advanced software such as a computer algebra system CAS.

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We are grateful to the many instructors from across the United States and Canada who have offered comments that assisted in the development and refinement of this book. These contributions included class testing, manuscript reviewing, exercise reviewing, and participating in surveys about the book and general course needs.

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Bob and Colin

INTRODUCTION TO CALCULUS

We begin with a brief introduction to some key ideas in calculus. It is not an exaggeration to say that calculus is one of the great intellectual achievements of humankind. Sending spacecraft to other planets, building computer systems for forecasting the weather, explaining the interactions between plants, insects, and animals, and understanding the structure of atoms are some of the countless scientific and technological advances that could not have been achieved without calculus. Moreover, calculus is a foundational part of the mathematical theory of analysis, a field that is under continuous development.

The primary formulation of calculus dates back to independent theories of Sir Isaac Newton and Gottfried Wilhelm Liebnitz in the 1600s. However, their work only remotely resembles the topics presented in this book. Through a few centuries of development and expansion, calculus has grown into the theory we present here. Newton and Liebnitz would likely be quite impressed that their calculus has evolved into a theory that many thousands of students around the world study each year.

There are two central concepts in calculus: the derivative and the integral. We introduce them next.

Maria Gaetana Agnesi (1718–1799), an Italian mathematician and theologian, is credited with writing one of the first books about calculus, *Instituzioni analitiche ad uso della gioventù italiana.* It was self-published and was written as a textbook for her brothers, who she was tutoring.

The Derivative The derivative of a function is simply the slope of its graph; it represents the rate of change of the function. For a linear function $y = 2.3x - 8.1$, the slope 2.3 indicates that y changes by 2.3 for each one-unit change in x. How do we find the slope of a graph of a function that is not linear, such as the one in Figure 1?

Imagine that this function represents the amount A of a drug in the bloodstream as a function of time t. Clearly, this situation is more complex than the linear case. The slope varies as we move along the curve. Initially positive because the amount of the drug in the bloodstream is increasing, the slope becomes negative as the drug is absorbed. Having an expression for the slope would enable us to know the time when the amount of the drug is a maximum (when the slope turns from positive to negative) or the time when the drug is leaving the bloodstream the fastest (a time to administer another dose).

To define the slope for a function that is not linear, we adapt the notion of slope for linear relationships. Specifically, to estimate the slope at point P in Figure 2, we select a point Q on the curve and draw a line between P and Q. We can use the slope of this line to approximate the slope at P. To improve this approximation, we move Q closer to P and calculate the slope of the new line. As Q moves closer to P, this approximation gets more precise. Although we cannot allow P and Q to be the same point (because we could no longer compute a slope), we instead "take the limit" of these slopes. We develop the concept of the limit in Chapter 2. Then in Chapter 3, we show that the limiting value may be defined as the exact slope at P.

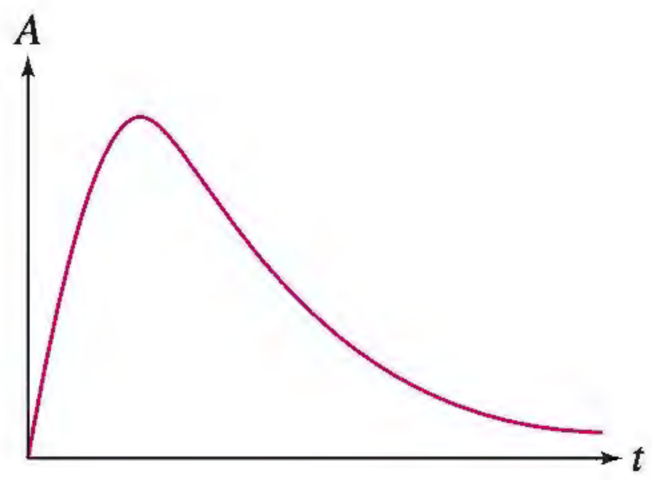

FIGURE 1

The Definite Integral The definite integral, another key calculus topic, can be thought of as adding up infinitely many infinitesimally small pieces of a whole. It too is obtained through a limiting process. More precisely, it is a limit of sums over a domain that is divided into progressively more and more pieces. To explore this idea, consider a solid

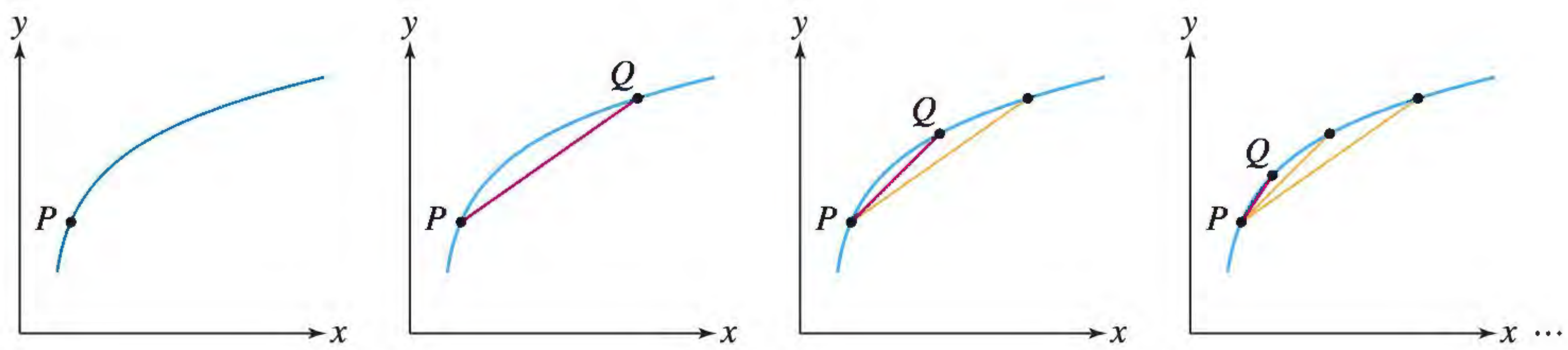

FIGURE 2 The approximation to the slope at P improves as Q approaches P.

ball of volume 2 cm^3 whose density (mass per unit volume) throughout is 1.5 g/cm^3. The mass of this ball is the product of density and volume, $(1.5)(2) = 3$ grams.

If the density is not the same throughout the ball (Figure 3), we can approximate its mass as follows:

- Chop the ball into a number of pieces,
- Assume the density is uniform on each piece and approximate the mass of each piece by multiplying density by volume,
- Add the approximate masses of the pieces to estimate the total mass of the ball.

We continually improve this approximation by chopping the ball into ever smaller pieces (Figure 4). Ultimately, an exact value is obtained by taking a limit of the approximate masses.

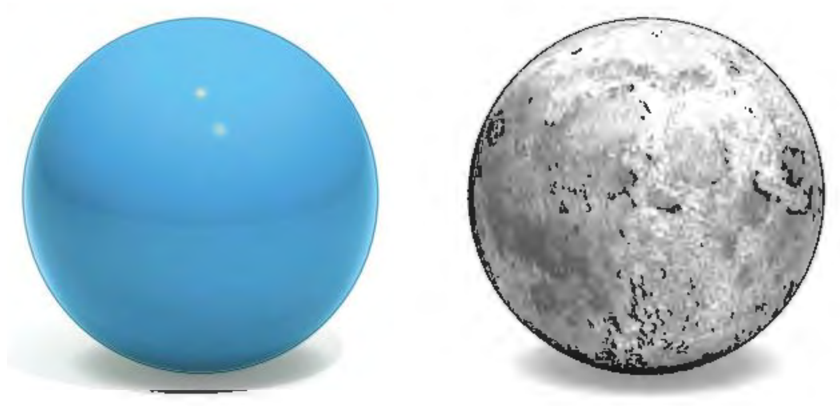

FIGURE 3 For a ball of uniform density, mass is the product of density and volume. For a nonuniform ball, a limiting process needs to be used to determine the mass.

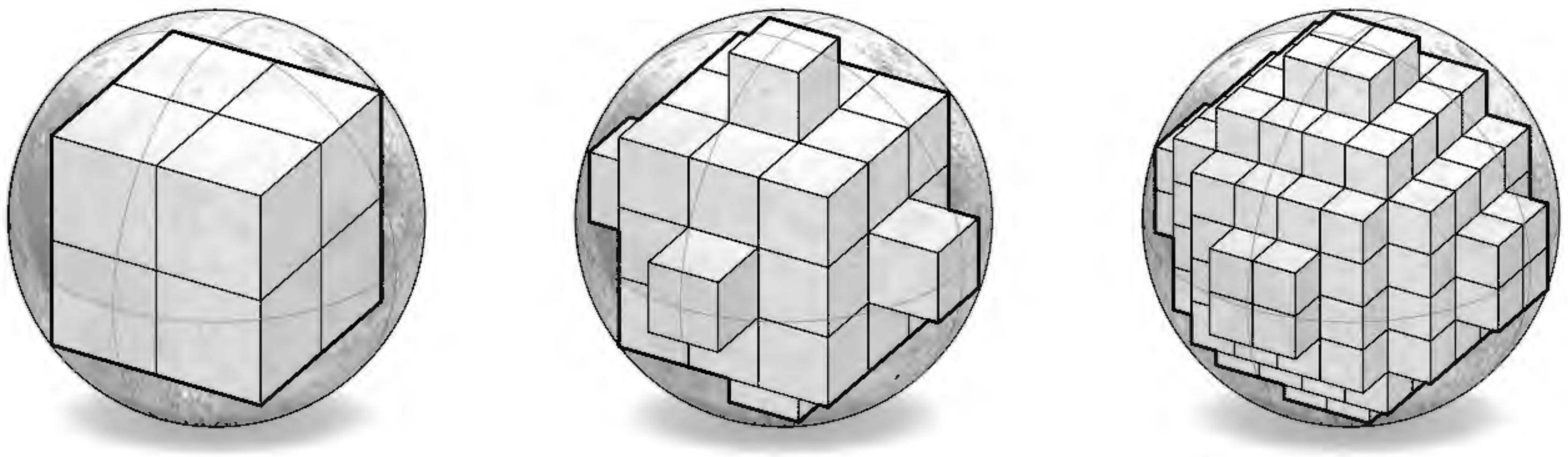

FIGURE 4

The irregular density of the moon presented a navigational challenge for spacecraft orbiting it. The first group of spacecraft (unmanned!) that circled the moon exhibited unexpected orbits. Space scientists realized that the density of the moon varied considerably and that the gravitational attraction of concentrations of mass (referred to as mascons) deflected the path of the spacecraft from the planned trajectory.

In Chapter 5, we define the definite integral in exactly this way; it is a limit of sums over an interval that is divided into progressively smaller subintervals.

The Fundamental Theorem of Calculus Although the derivative and the definite integral are very different concepts, it turns out they are related through an important theorem called the Fundamental Theorem of Calculus presented in Chapter 5. This theorem demonstrates that the derivative and the definite integral are, to some extent, inverses of each other, a relationship that we will find beneficial in many ways.

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Oscillatory phenomena, such as the extreme tides of the Bay of Fundy in Atlantic Canada, are modeled by the sine function from trigonometry.

1 PRECALCULUS REVIEW

Calculus builds on the foundation of algebra, analytic geometry, and trigonometry. In this chapter, therefore, we review some concepts, facts, and formulas from precalculus that are used throughout the text. In the last section, we discuss ways in which technology can be used to enhance your visual understanding of functions and their properties.

1.1 Real Numbers, Functions, and Graphs

We begin with a short discussion of real numbers. This gives us the opportunity to recall some basic properties and standard notation.

A **real number** is a number represented by a decimal or "decimal expansion." There are three types of decimal expansions: finite, infinite repeating, and infinite but nonrepeating. For example,

$$\frac{3}{8} = 0.375, \qquad \frac{1}{7} = 0.142857142857\ldots = 0.\overline{142857} \qquad \pi = 3.141592653589793\ldots$$

The number $\frac{3}{8}$ is represented by a finite decimal, whereas $\frac{1}{7}$ is represented by an infinite repeating decimal. The bar over 142857 indicates that this sequence repeats indefinitely. The decimal expansion of π is infinite but nonrepeating.

The set of all real numbers is denoted by a boldface **R**. When there is no risk of confusion, we refer to a real number simply as a *number*. We also use the standard symbol $\in$ for the phrase "belongs to." Thus,

$$a \in \mathbf{R} \qquad \text{reads} \qquad \text{"}a \text{ belongs to } \mathbf{R}\text{"}$$

The set of integers is commonly denoted by the letter **Z** (this choice comes from the German word *Zahl*, meaning "number"). Thus, $\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. A **whole number** is a nonnegative integer—that is, one of the numbers $0, 1, 2, \ldots$.

A real number is called **rational** if it can be represented by a fraction p/q, where p and q are integers with $q \neq 0$. The set of rational numbers is denoted **Q** (for "quotient"). Numbers that are not rational, such as π and $\sqrt{2}$, are called **irrational**.

We can tell whether a number is rational from its decimal expansion: Rational numbers have finite or infinite repeating decimal expansions, and irrational numbers have infinite, nonrepeating decimal expansions. Furthermore, the decimal expansion of a number is unique, apart from the following exception: Every finite decimal is equal to an infinite decimal in which the digit 9 repeats. For example, $1/5 = 0.5 = 0.499999\ldots$

Two algebraic properties of the real numbers are the commutative property of addition, $a + b = b + a$, and the distributive property of multiplication over addition, $a(b + c) = ab + ac$. A list of further properties can be found in Appendix B. Next, we present some properties of exponents that are used regularly when we work with exponential expressions and functions.

	Rule	Example
Exponent zero	$b^0 = 1$	$5^0 = 1$
Products	$b^x b^y = b^{x+y}$	$2^5 \cdot 2^3 = 2^{5+3} = 2^8$
Quotients	$\frac{b^x}{b^y} = b^{x-y}$	$\frac{4^7}{4^2} = 4^{7-2} = 4^5$
Negative exponents	$b^{-x} = \frac{1}{b^x}$	$3^{-4} = \frac{1}{3^4}$
Power to a power	$\left(b^x\right)^y = b^{xy}$	$\left(3^2\right)^4 = 3^{2(4)} = 3^8$
Roots	$b^{1/n} = \sqrt[n]{b}$	$5^{1/2} = \sqrt{5}$

EXAMPLE 1 Rewrite as a whole number or fraction:

(a) $16^{-1/2}$ **(b)** $27^{2/3}$ **(c)** $4^{16} \cdot 4^{-18}$ **(d)** $\dfrac{9^3}{3^7}$

Solution

(a) $16^{-1/2} = \dfrac{1}{16^{1/2}} = \dfrac{1}{\sqrt{16}} = \dfrac{1}{4}$

(b) $27^{2/3} = \left(27^{1/3}\right)^2 = 3^2 = 9$

(c) $4^{16} \cdot 4^{-18} = 4^{-2} = \dfrac{1}{4^2} = \dfrac{1}{16}$

(d) $\dfrac{9^3}{3^7} = \dfrac{\left(3^2\right)^3}{3^7} = \dfrac{3^6}{3^7} = 3^{-1} = \dfrac{1}{3}$ ■

Another important algebraic relationship is the binomial expansion of $(a+b)^n$. It is proved in Appendix C and is needed in the proof of the power law for derivatives in Section 3.2.

Expanding $(a+b)^n$ for $n = 2, 3, 4$, we obtain

- $(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2$
- $(a+b)^3 = (a+b)(a+b)^2 = (a+b)(a^2+2ab+b^2) = a^3 + 3a^2b + 3ab^2 + b^3$
- $(a+b)^4 = (a+b)(a+b)^3 = (a+b)(a^3+3a^2b+3ab^2+b^3) = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + a^4$

Notice there are some patterns emerging here. In each case, the first and second terms are a^n and $na^{n-1}b$, while the last two terms are nab^{n-1} and b^n. There is a general formula for the expansion, called the **binomial expansion formula**. It is expressed using summation notation as

REMINDER n*-factorial is the number*

$$n! = n(n-1)(n-2)\cdots(2)(1)$$

Thus,

$$1! = 1, \quad 2! = (2)(1) = 2$$

$$3! = (3)(2)(1) = 6$$

By convention, we set $0! = 1$.

$$(a+b)^n = \sum_{p=0}^{n} \frac{n!}{(n-p)!\,p!} a^{n-p} b^p$$

We introduce summation notation in Section 5.1. For now, you can understand the formula as saying that $(a+b)^n$ is a sum of terms $\frac{n!}{(n-p)!\,p!}a^{n-p}b^p$, with a term for each p going from 0 to n. So, for example, in $(a+b)^8$, the first four terms are: $\frac{8!}{8!0!}a^8 = a^8$, $\frac{8!}{7!1!}a^7b = 8a^7b$, $\frac{8!}{6!2!}a^6b^2 = 28a^6b^2$, and $\frac{8!}{5!3!}a^5b^3 = 56a^5b^3$. Working out the rest of the terms, we find that:

$$(a+b)^8 = a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + a^8$$

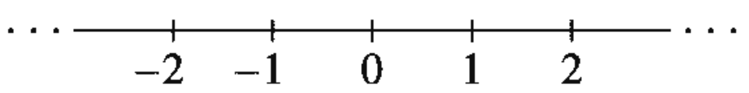


FIGURE 1 The set of real numbers represented as a line.

We visualize real numbers as points on a line (Figure 1), and we refer to that line as the **real line**. For this reason, real numbers are often called **points**. The point corresponding to 0 is called the **origin**.

The real numbers are ordered, and we can view that ordering in terms of position on the real line: p is **greater than** q, written $p > q$, if p is to the right of q on the number line. p is **less than** q, written $p < q$, if p is to the left of q on the number line.

A real number x is said to be **positive** if $x > 0$, **negative** if $x < 0$, **nonpositive** if $x \le 0$, and **nonnegative** if $x \ge 0$.

In some texts, "larger than" is used synonymously with "greater than." We will avoid that usage in this text.

Two other important terms we use, related to position on the real line, are "large" and "small." We say that p is **large** if p is distant from the origin, and p is **small** if p is close to the origin. While these definitions are somewhat vague, the meaning should be clear in the contexts in which they are used.

The **absolute value** of a real number a, denoted $|a|$, is defined by (Figure 2):

$$|a| = \text{distance from the origin} = \begin{cases} a & \text{if } a \ge 0 \\ -a & \text{if } a < 0 \end{cases}$$

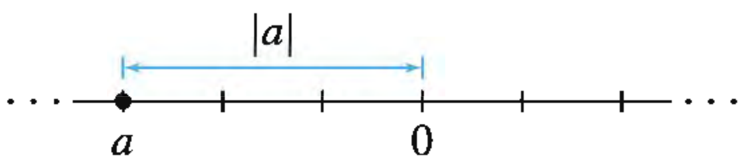


FIGURE 2 $|a|$ is the distance from a to the origin.

For example, $|1.2| = 1.2$ and $|-8.35| = -(-8.35) = 8.35$. The absolute value satisfies

$$\boxed{|a| = |-a|, \qquad |ab| = |a|\,|b|}$$

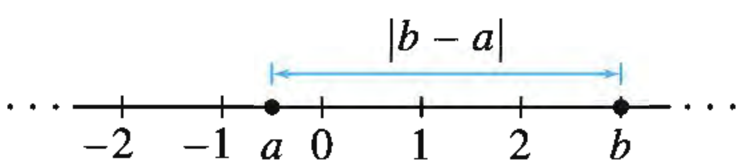


FIGURE 3 The distance between a and b is $|b-a|$.

The **distance** between two real numbers a and b is $|b - a|$, which is the length of the line segment joining a and b (Figure 3).

Two real numbers a and b are close to each other if $|b - a|$ is small, and this is the case if their decimal expansions agree to many places. More precisely, *if the decimal expansions of a and b agree to k places (to the right of the decimal point), then the distance $|b - a|$ is at most 10^{-k}*. Thus, the distance between $a = 3.1415$ and $b = 3.1478$ is at most 10^{-2} because a and b agree to two places. In fact, the distance is exactly $|3.1478 - 3.1415| = 0.0063$.

Beware that $|a + b|$ is not equal to $|a| + |b|$ unless a and b have the same sign or at least one of a and b is zero. If they have opposite signs, cancellation occurs in the sum $a + b$, and $|a + b| < |a| + |b|$. For example, $|2 + 5| = |2| + |5|$ but $|-2 + 5| = 3$, which is less than $|-2| + |5| = 7$. In any case, $|a + b|$ is never greater than $|a| + |b|$, and this gives us the simple but important **triangle inequality**:

$$\boxed{|a + b| \le |a| + |b|} \tag{1}$$

We use standard notation for intervals. Given real numbers $a < b$, there are four intervals with endpoints a and b (Figure 4). They all have length $b - a$ but differ according to which endpoints are included.

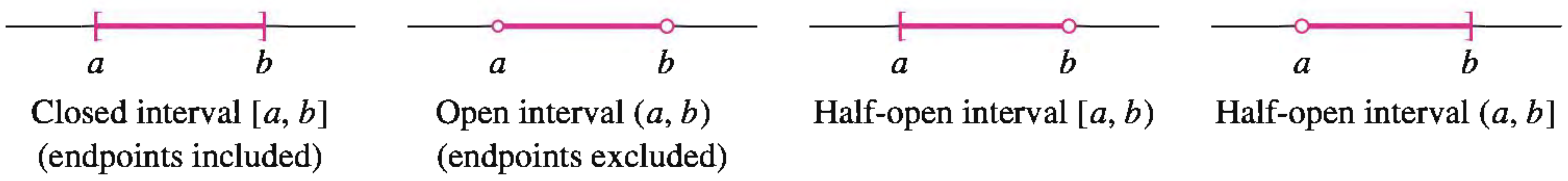


FIGURE 4 The four intervals with endpoints a and b.

The **closed interval** $[a, b]$ is the set of all real numbers x such that $a \le x \le b$:

$$[a, b] = \{x \in \mathbf{R} : a \le x \le b\}$$

We usually write this more simply as $\{x : a \le x \le b\}$, it being understood that x belongs to $\mathbf{R}$. The **open** and **half-open intervals** are the sets

$$\underbrace{(a, b) = \{x : a < x < b\}}_{\text{Open interval (endpoints excluded)}}, \qquad \underbrace{[a, b) = \{x : a \le x < b\}}_{\text{Half-open interval}}, \qquad \underbrace{(a, b] = \{x : a < x \le b\}}_{\text{Half-open interval}}$$

The infinite interval $(-\infty, \infty)$ is the entire real line $\mathbf{R}$. A half-infinite interval is closed if it contains its finite endpoint and is open otherwise (Figure 5):

$$[a, \infty) = \{x : x \ge a\}, \qquad (-\infty, b] = \{x : x \le b\}$$

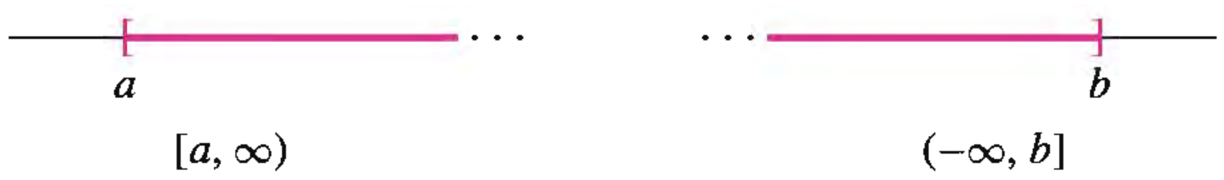


FIGURE 5 Closed half-infinite intervals.

Open and closed intervals may be described by absolute-value inequalities. For example, the interval $(-r, r)$ is described by the inequality $|x| < r$ (Figure 6):

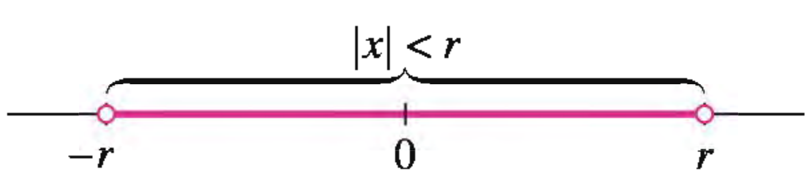


FIGURE 6 The interval $(-r, r) = \{x : |x| < r\}$.

$$\boxed{|x| < r \quad \Leftrightarrow \quad -r < x < r \quad \Leftrightarrow \quad x \in (-r, r)} \tag{2}$$

The symbol $\Leftrightarrow$ is read as "is equivalent to," and the symbol $\Rightarrow$, that we will also use, is read as "implies."

More generally, for an interval symmetric about the value c (Figure 7):

$$\boxed{|x - c| < r \quad \Leftrightarrow \quad c - r < x < c + r \quad \Leftrightarrow \quad x \in (c - r, c + r)} \qquad \boxed{3}$$

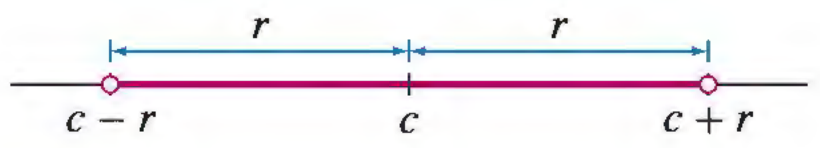


FIGURE 7 $(a, b) = (c - r, c + r)$.

Closed intervals can be represented similarly, with $<$ replaced by $\le$. We refer to r as the **radius** and c as the **midpoint** or **center** of the intervals $(c - r, c + r)$ and $[c - r, c + r]$. The intervals (a, b) and $[a, b]$ have midpoint $c = \frac{1}{2}(a + b)$ and radius $r = \frac{1}{2}(b - a)$ (Figure 7).

EXAMPLE 2 Describe $[7, 13]$ using an absolute-value inequality.

Solution The midpoint of the interval $[7, 13]$ is $c = \frac{1}{2}(7 + 13) = 10$, and its radius is $r = \frac{1}{2}(13 - 7) = 3$ (Figure 8). Therefore,

$$[7, 13] = \{x \in \mathbf{R} : |x - 10| \le 3\}$$

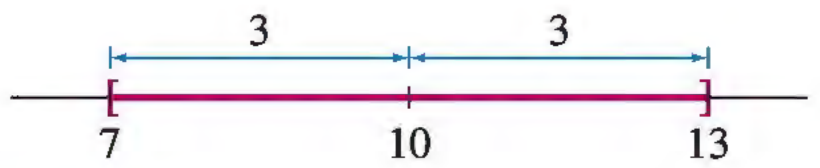


FIGURE 8 The interval $[7, 13]$ is described by $|x - 10| \le 3$.

EXAMPLE 3 Describe the set $S = \left\{x : \left|\frac{1}{2}x - 3\right| > 4\right\}$ in terms of intervals.

Solution It is easier to consider the opposite inequality $\left|\frac{1}{2}x - 3\right| \le 4$ first. By (2):

In Example 3, we use the notation $\cup$ to denote "union": The union $A \cup B$ of sets A and B consists of all elements that belong to either A or B (or to both).

$$\begin{aligned}
\left|\frac{1}{2}x - 3\right| \le 4 \quad &\Leftrightarrow \quad -4 \le \frac{1}{2}x - 3 \le 4 \\
& \quad -1 \le \frac{1}{2}x \le 7 \qquad \text{(add 3)} \\
& \quad -2 \le x \le 14 \qquad \text{(multiply by 2)}
\end{aligned}$$

FIGURE 9 The set $S = \left\{x : \left|\frac{1}{2}x - 3\right| > 4\right\}$.

Thus, $\left|\frac{1}{2}x - 3\right| \le 4$ is satisfied when x belongs to $[-2, 14]$. The set S is the *complement*, consisting of all numbers x *not in* $[-2, 14]$. We can describe S as the union of two intervals: $S = (-\infty, -2) \cup (14, \infty)$ (Figure 9).

Graphing

The term "Cartesian" refers to the French philosopher and mathematician René Descartes (1596–1650), whose Latin name was Cartesius. He is credited (along with Pierre de Fermat) with the invention of analytic geometry. In his great work La Géométrie, *Descartes used the letters x, y, z for unknowns and a, b, c for constants, a convention that has been followed ever since.*

Graphing is a basic tool in calculus, as it is in algebra and trigonometry. Recall that rectangular (or Cartesian) coordinates in the plane are defined by choosing two perpendicular axes, the x-axis and the y-axis.To a pair of numbers (a, b) we associate the point P located at the intersection of the line perpendicular to the x-axis at a and the line perpendicular to the y-axis at b [Figure 10(A)]. The numbers a and b are the x- and y-**coordinates** of P. The x-coordinate is sometimes called the **abscissa** and the y-coordinate the **ordinate**. The **origin** is the point with coordinates $(0, 0)$.

The notation (a, b) could mean the open interval that is equal to the set of points $\{x : a < x < b\}$ or it could mean the point in the xy-plane with $x = a$ and $y = b$. In general, the meaning will be apparent from the context.

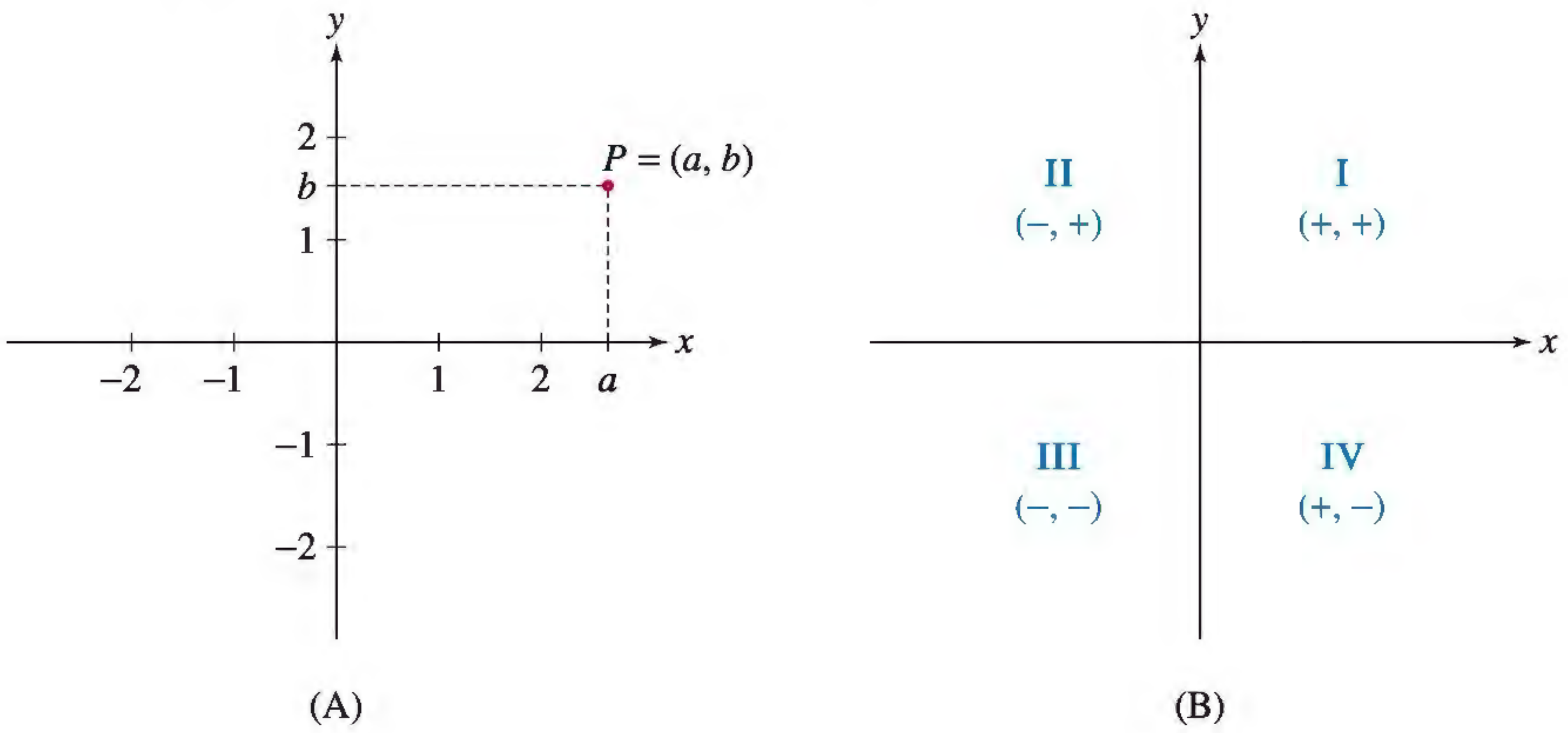


FIGURE 10 The rectangular coordinate system (A) and the four quadrants (B).

The axes divide the plane into four quadrants labeled I–IV, determined by the signs of the coordinates [Figure 10(B)]. For example, quadrant III consists of points (x, y) such that $x < 0$ and $y < 0$.

The distance d between two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ is computed using the Pythagorean Theorem. In Figure 11, we see that $\overline{P_1P_2}$ is the hypotenuse of a right triangle with sides $a = |x_2 - x_1|$ and $b = |y_2 - y_1|$. Therefore,

$$d^2 = a^2 + b^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

We obtain the distance formula by taking square roots.

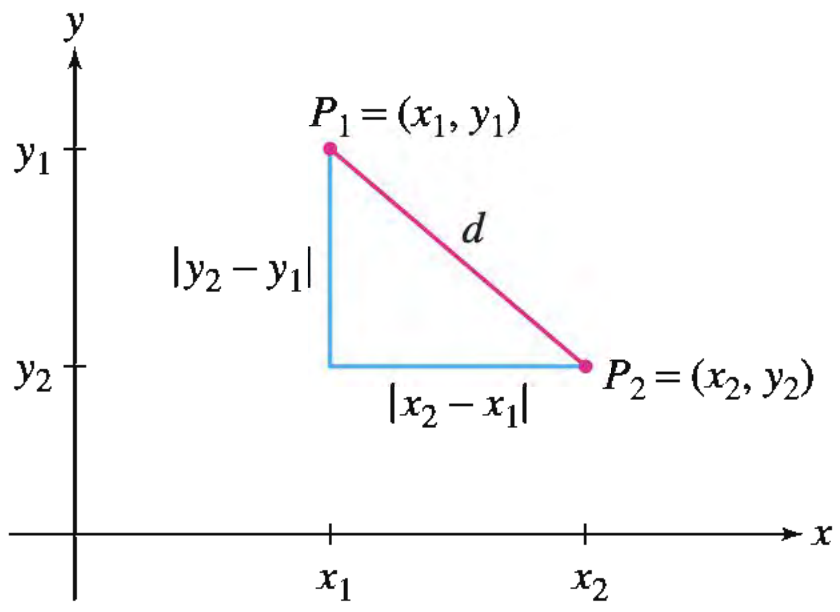


FIGURE 11 Distance d is given by the distance formula.

Distance Formula The distance between $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ is

$$\boxed{d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

Once we have the distance formula, we can derive the equation of a circle of radius r and center (a, b) (Figure 12). A point (x, y) lies on this circle if the distance from (x, y) to (a, b) is r:

$$\sqrt{(x - a)^2 + (y - b)^2} = r$$

Squaring both sides, we obtain the standard equation of the circle of radius r centered at (a, b):

$$\boxed{(x - a)^2 + (y - b)^2 = r^2}$$

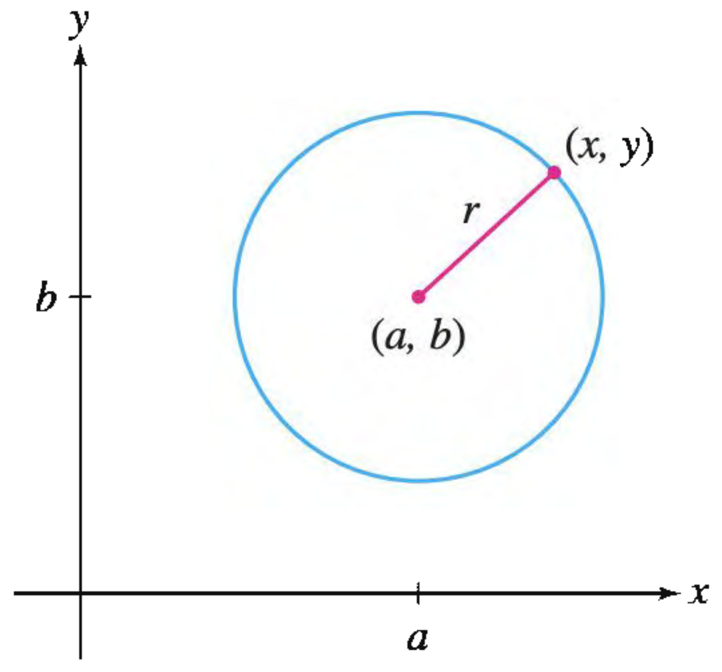


FIGURE 12 Circle with equation $(x - a)^2 + (y - b)^2 = r^2$.

We now review some definitions and notation concerning functions.

DEFINITION A **function** f from a set D to a set Y is a rule that assigns, to each element x in D, a unique element $y = f(x)$ in Y. We write

$$f : D \to Y$$

*A function $f : D \to Y$ is also called a **map**. The sets D and Y can be arbitrary. For example, we can define a map from the set of living people to the set of whole numbers by mapping each person to his or her year of birth. The range of this map is the set of years in which a living person was born. In multivariable calculus, the domain might be a set of points in the two-dimensional plane and the range a set of numbers, points, or vectors.*

The set D, called the **domain** of f, is the set of "allowable inputs." For $x \in D$, $f(x)$ is called the **value** of f at x (Figure 13). The **range** R of f is the subset of Y consisting of all values $f(x)$:

$$R = \{y \in Y : f(x) = y \text{ for some } x \in D\}$$

Informally, we think of f as a "machine" that produces an output y for every input x in the domain D (Figure 14).

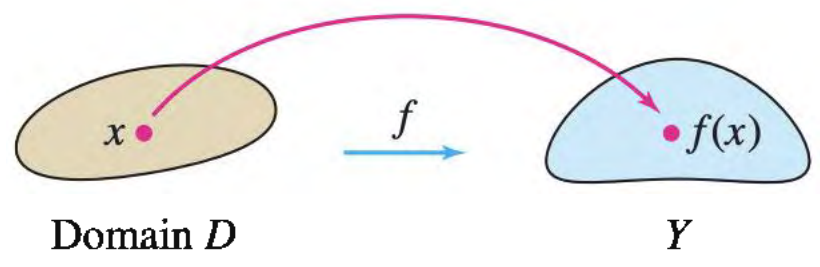


FIGURE 13 A function assigns an element $f(x)$ in Y to each $x \in D$.

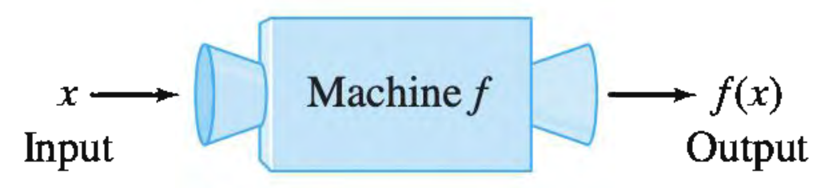


FIGURE 14 Think of f as a "machine" that takes the input x and produces the output $f(x)$.

Writing $y = f(x)$ for a function f, we refer to x as the **independent variable** and y as the **dependent variable** (because its value depends on the choice of x).

The first part of this text deals with functions f, where both the domain and the range are sets of real numbers. When f is defined by a formula, its natural domain is the set of real numbers x for which the formula is meaningful. For example, the function $f(x) = \sqrt{9 - x}$ has domain $D = \{x : x \le 9\}$ because $\sqrt{9 - x}$ is defined if $9 - x \ge 0$. Here are some other examples of domains and ranges:

$f(x)$	**Domain** D	**Range** R
x^2	**R**	$\{y : y \ge 0\}$
$\cos x$	**R**	$\{y : -1 \le y \le 1\}$
$\frac{1}{x+1}$	$\{x : x \ne -1\}$	$\{y : y \ne 0\}$

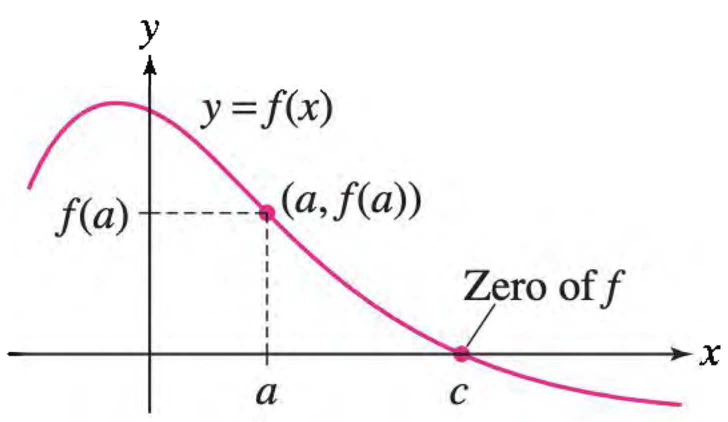


FIGURE 15

The **graph** of a function $y = f(x)$ is obtained by plotting the points $(a, f(a))$ for a in the domain D (Figure 15). If you start at $x = a$ on the x-axis, and move up to the graph and then over to the y-axis, you arrive at the value $f(a)$.

A **zero** or **root** of a function f is a number c such that $f(c) = 0$. The zeros are the values of x where the graph intersects the x-axis.

In Chapter 4, we will use calculus to sketch and analyze graphs. At this stage, to sketch a graph by hand, we can make a table of function values, plot the corresponding points (including any zeros), and connect them by a smooth curve.

EXAMPLE 4 Find the roots and sketch the graph of $f(x) = x^3 - 2x$.

Solution First, we solve

$$x^3 - 2x = x(x^2 - 2) = 0$$

The roots of f are $x = 0$ and $x = \pm\sqrt{2}$. To sketch the graph, we plot the roots and a few values listed in Table 1 and join them by a curve (Figure 16). ■

TABLE 1

x	$x^3 - 2x$
−2	−4
−1	1
0	0
1	−1
2	4

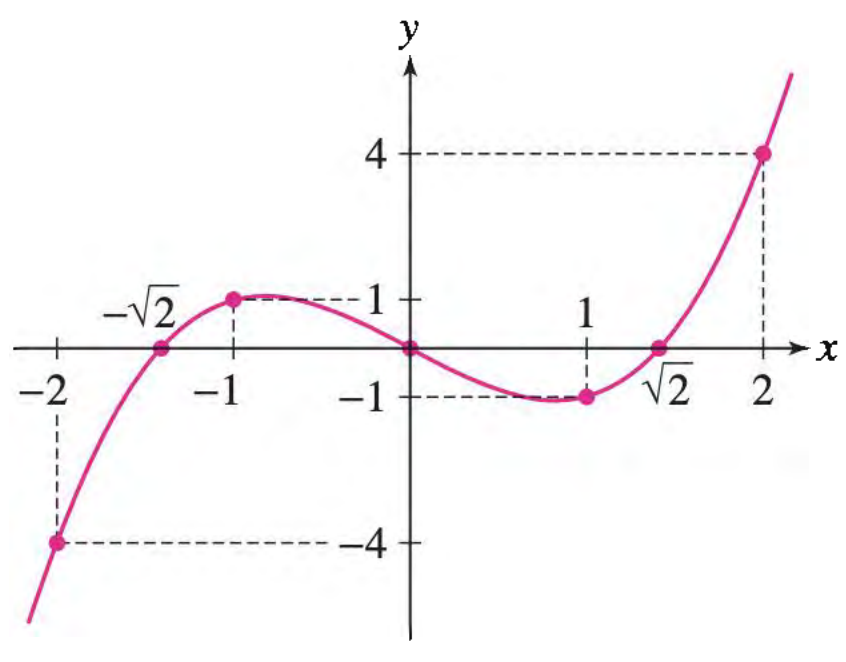


FIGURE 16 Graph of $f(x) = x^3 - 2x$.

Harvepino/Shutterstock

FIGURE 17 The Greenland ice sheet.

Functions arising in applications are not always given by formulas. Data collected from observation or experiment define functions for which there may be no exact formula. Such functions can be displayed either graphically or by a table of values. For example, consider the mass of the Greenland ice sheet (Figure 17) that covers most of the island of Greenland. Data in Table 2 and Figure 18 collected by NASA's GRACE (Global Recovery and Climate Experiment) satellite show the change in the mass of the ice, C, as a function of time, t, since the beginning of 2012. (Note, for example, $t = 1.46$ means 0.46 years into 2013.) To plot this function, we plot the data points in the table and connect the points with a smooth curve. We will see that many of the tools of calculus can be applied to functions constructed from data in this way.

TABLE 2

Time (years since Jan. 1, 2012)	Change in Mass from Jan. 1, 2012 (in gigatonnes)	Time (years since Jan. 1, 2012)	Change in Mass from Jan. 1, 2012 (in gigatonnes)
0	0	2.79	−794.51
0.21	138.53	3.12	−623.4
0.54	−139.14	3.32	−624.41
0.89	−487.05	3.70	−960.08
1.12	−386.78	4.12	−899.86
1.46	−355.26	4.46	−869.46
1.87	−518.52	4.91	−1153.08
2.21	−475.14	5.25	−1110.29
2.45	−474.96	5.44	−1115.94

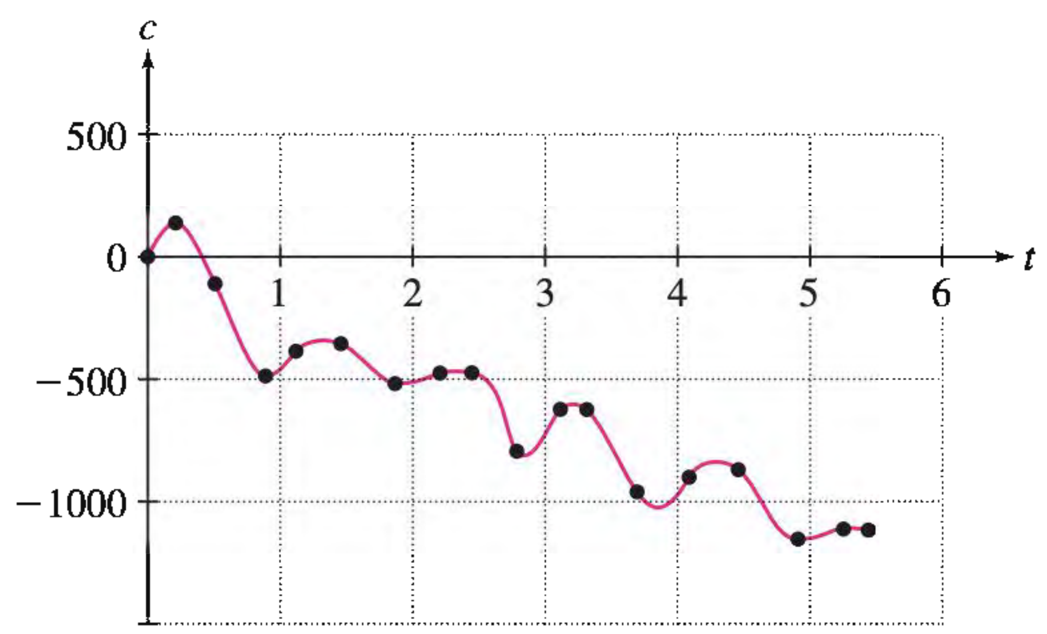


FIGURE 18 Change in mass of the Greenland ice sheet.

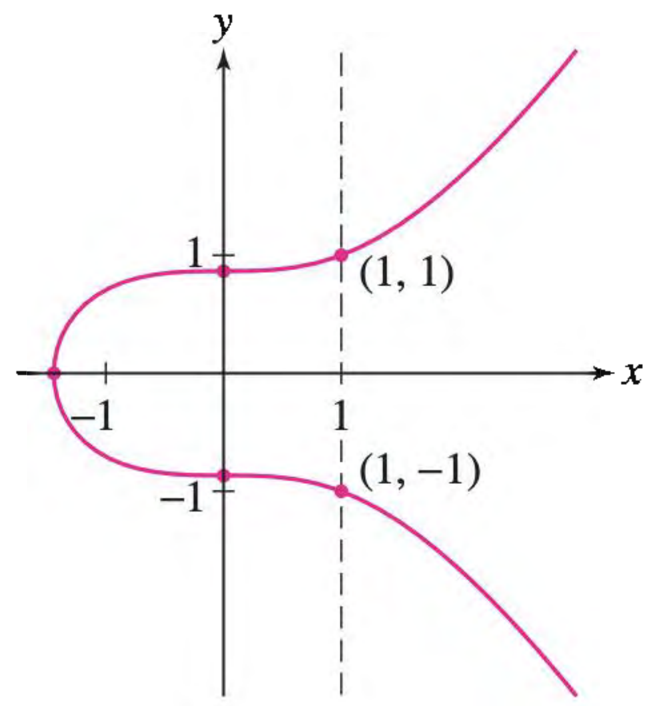


FIGURE 19 Graph of $4y^2 - x^3 = 3$. This graph fails the Vertical Line Test, so it is not the graph of a function.

We can graph not just functions but, more generally, any equation relating y and x. Figure 19 shows the graph of the equation $4y^2 - x^3 = 3$; it consists of all pairs (x, y) satisfying the equation. This curve is not the graph of a function of x because some x-values are associated with two y-values. For example, $x = 1$ is associated with both $y = 1$ and $y = -1$. A curve is the graph of a function of x if and only if it passes the **Vertical Line Test**; that is, every vertical line $x = a$ intersects the curve in at most one point.

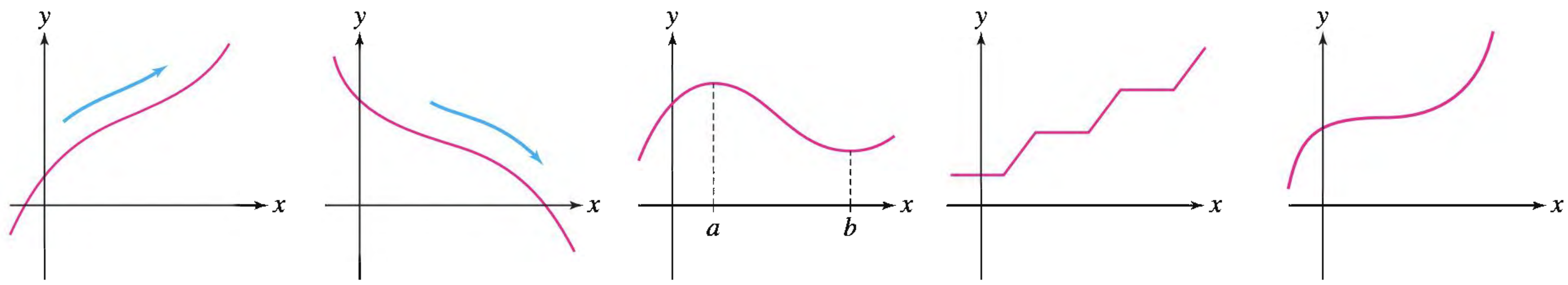
We are often interested in whether a function is increasing or decreasing. Roughly speaking, a function f is increasing if its graph goes up as we move to the right and is decreasing if its graph goes down [Figures 20(A) and (B)]. More precisely, we define the notion of increase/decrease on an open interval.

A function f is

- **Increasing** on (a, b) if $f(x_1) < f(x_2)$ for all $x_1, x_2 \in (a, b)$ such that $x_1 < x_2$
- **Decreasing** on (a, b) if $f(x_1) > f(x_2)$ for all $x_1, x_2 \in (a, b)$ such that $x_1 < x_2$

We say that f is **monotonic** if it is either increasing or decreasing. In Figure 20(C), the function is not monotonic because, while it is increasing for some intervals of x and decreasing for others, it is neither increasing nor decreasing for all x.

A function f is called **nondecreasing** if $f(x_1) \leq f(x_2)$ for $x_1 < x_2$ (defined by $\leq$ rather than a strict inequality $<$). **Nonincreasing** functions are defined similarly. Function (D) in Figure 20 is nondecreasing, but it is not increasing on the intervals where the graph is horizontal. Function (E) is increasing everywhere, even though it levels off momentarily.



(A) Increasing (B) Decreasing (C) Decreasing on (a, b) but not decreasing everywhere (D) Nondecreasing but not increasing (E) Increasing

FIGURE 20

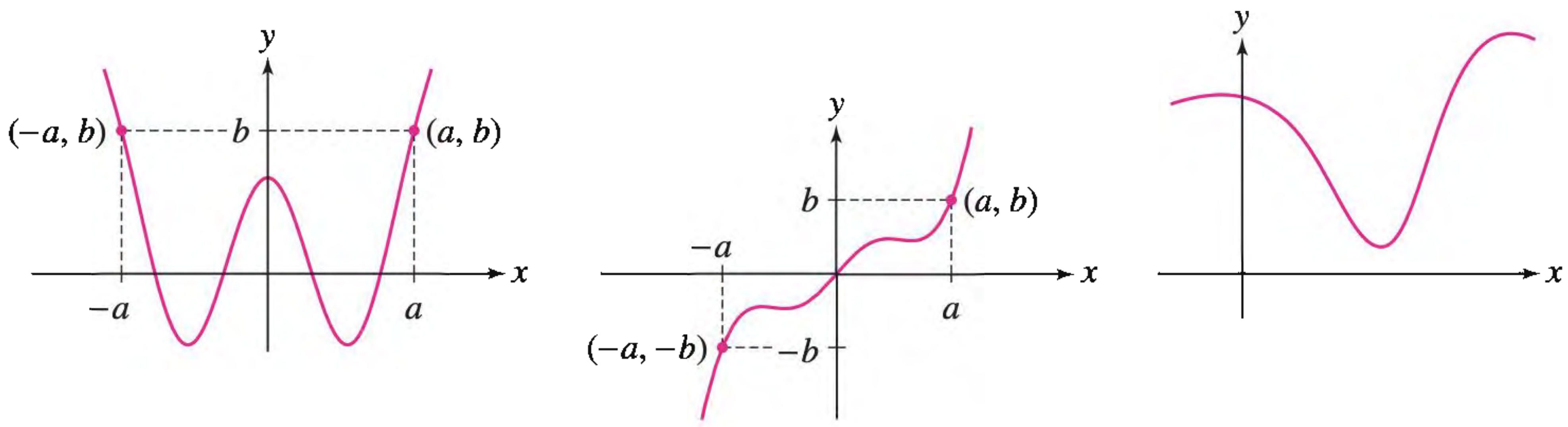
Another important property of functions is **parity**, which refers to whether a function is even or odd:

- f is **even** if $\quad f(-x) = f(x)$.
- f is **odd** if $\quad f(-x) = -f(x)$.

The graphs of functions with even or odd parity have a special symmetry:

- **Even function:** The graph is symmetric about the y-axis. This means that if $P = (a, b)$ lies on the graph, then so does $Q = (-a, b)$ [Figure 21(A)].
- **Odd function:** The graph is symmetric with respect to the origin. This means that if $P = (a, b)$ lies on the graph, then so does $Q = (-a, -b)$ [Figure 21(B)].

Many functions are neither even nor odd [Figure 21(C)].



(A) Even function: $f(-x) = f(x)$
Graph is symmetric about the y-axis.

(B) Odd function: $f(-x) = -f(x)$
Graph is symmetric about the origin.

(C) Neither even nor odd

FIGURE 21

EXAMPLE 5 Determine whether the function is even, odd, or neither.

(a) $f(x) = x^4$ **(b)** $g(x) = x^{-1}$ **(c)** $h(x) = x^2 + x$

Solution

(a) $f(-x) = (-x)^4 = x^4$. Thus, $f(x) = f(-x)$, and f is even.
(b) $g(-x) = (-x)^{-1} = -x^{-1}$. Thus, $g(-x) = -g(x)$, and g is odd.
(c) $h(-x) = (-x)^2 + (-x) = x^2 - x$. We see that $h(-x)$ is not equal to $h(x)$ or to $-h(x) = -x^2 - x$. Therefore, h is neither even nor odd. ■

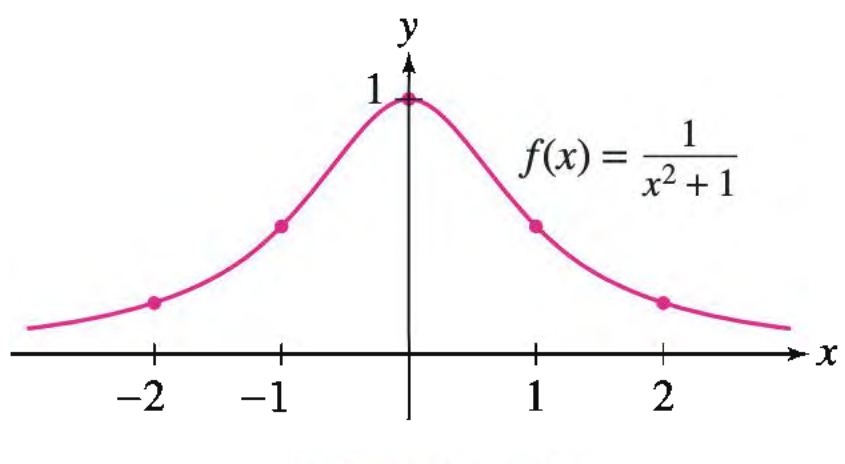
EXAMPLE 6 **Using Symmetry** Sketch the graph of $f(x) = \dfrac{1}{x^2+1}$.

Solution The function f is positive $[f(x) > 0]$ and even $[f(-x) = f(x)]$. Therefore, the graph lies above the x-axis and is symmetric with respect to the y-axis.

Furthermore, f is decreasing for $x \geq 0$ (because a larger value of x makes the denominator larger and therefore the fraction smaller). We use this information and a short table of values (Table 3) to sketch the graph (Figure 22). Note that the graph approaches the x-axis as we move away from zero, both to the right and to the left, because $f(x)$ gets closer to zero as $|x|$ increases. ■

TABLE 3

x	$\dfrac{1}{x^2+1}$
0	1
± 1	$\frac{1}{2}$
± 2	$\frac{1}{5}$



DF **FIGURE 22**

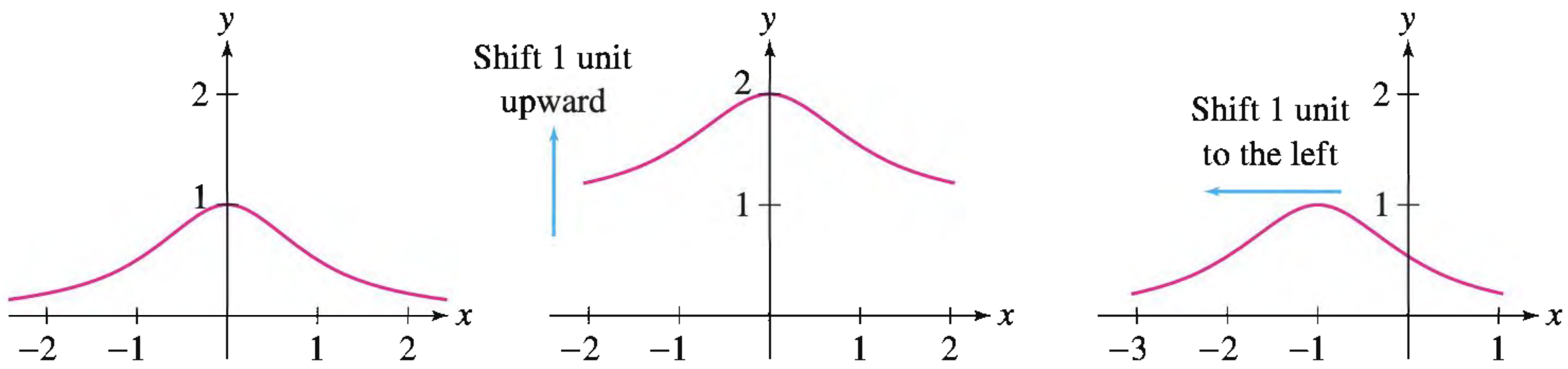
Two important ways of modifying a graph are **translation** (or **shifting**) and **scaling**. Translation consists of moving the graph horizontally or vertically:

Remember that $f(x)+c$ and $f(x+c)$ are different. The graph of $y = f(x)+c$ is a vertical translation and $y = f(x+c)$ is a horizontal translation of the graph of $y = f(x)$.

DEFINITION Translation (Shifting)

- **Vertical Translation** $y = f(x) + c$: Shifts the graph of f by $|c|$ units *vertically*, upward if $c > 0$ and downward if $c < 0$.
- **Horizontal Translation** $y = f(x + c)$: Shifts the graph of f by $|c|$ units *horizontally*, to the right if $c < 0$ and to the left if $c > 0$.

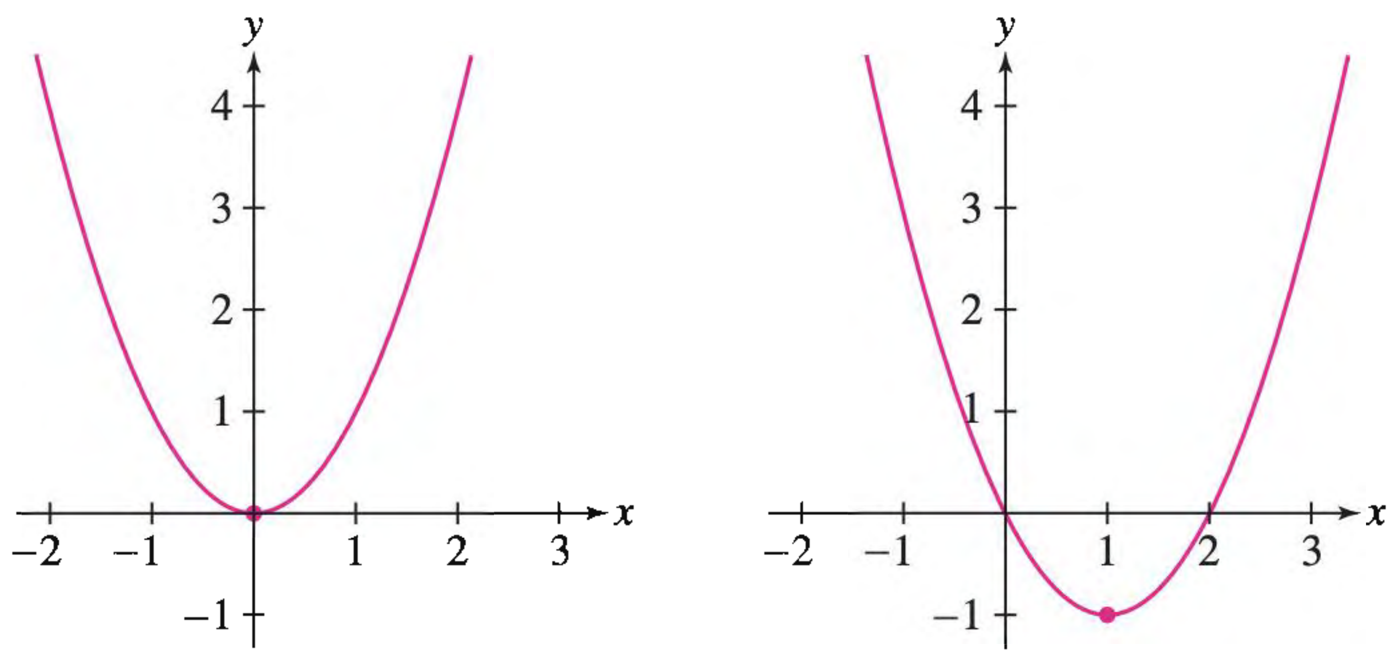
Figure 23 shows the effect of translating the graph of $f(x) = 1/(x^2 + 1)$ vertically and horizontally.



(A) $y = f(x) = \dfrac{1}{x^2+1}$ (B) $y = f(x) + 1 = \dfrac{1}{x^2+1} + 1$ (C) $y = f(x+1) = \dfrac{1}{(x+1)^2+1}$

FIGURE 23

EXAMPLE 7 Figure 24(A) is the graph of $f(x) = x^2$, and Figure 24(B) is a horizontal and vertical shift of (A). What is the equation of graph (B)?



(A) $f(x) = x^2$ (B)

DF **FIGURE 24**

Solution Graph (B) is obtained by shifting graph (A) 1 unit to the right and 1 unit down. We can see this by observing that the point $(0, 0)$ on the graph of f is shifted to $(1, -1)$. Therefore, (B) is the graph of $g(x) = (x - 1)^2 - 1$. ■

Scaling (also called **dilation**) consists of compressing or expanding the graph in the vertical or horizontal directions:

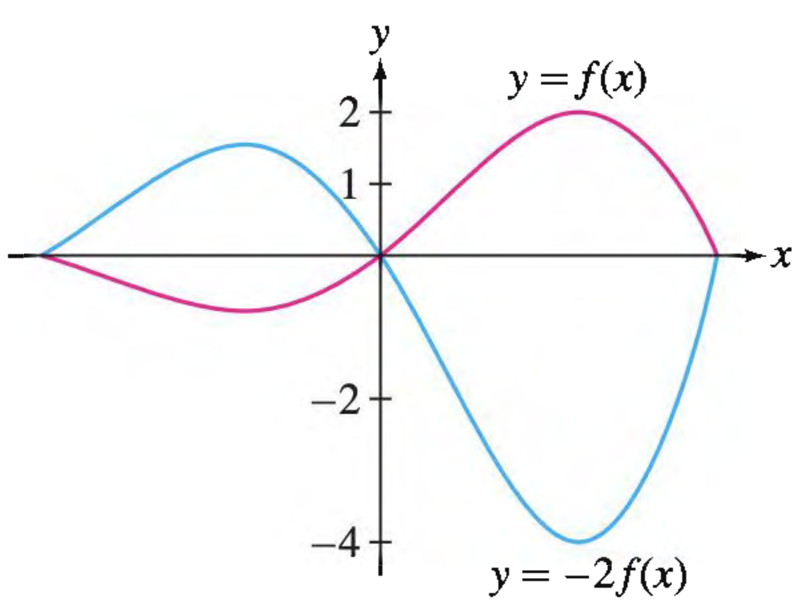


FIGURE 25 Negative vertical scale factor $k = -2$.

DEFINITION Scaling

- **Vertical scaling** $y = kf(x)$: If $|k| > 1$, the graph of f is expanded vertically by the factor $|k|$. If $0 < |k| < 1$, the graph of f is compressed vertically by the factor $|k|$. If $k < 0$, then the graph is also reflected across the x-axis (Figure 25).
- **Horizontal scaling** $y = f(kx)$: If $|k| > 1$, the graph of f is compressed horizontally by the factor $|k|$. If $0 < |k| < 1$, the graph of f is expanded horizontally by the factor $|k|$. If $k < 0$, then the graph is also reflected across the y-axis.